Scaling Limit of Vicious Walkers, Gaussian Random Matrix Ensembles, and Dyson Brownian Motions

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(Dated: February 1, 2008)

We study systems of interacting Brownian particles in one dimension constructed as the diffusion scaling limits of Fisher's vicious walk models. We define two types of nonintersecting Brownian motions, in which we impose no condition (resp. nonintersecting condition forever) in the future for the first-type (resp. second-type). It is shown that, when all particles start from the origin, their positions at time 1 in the first-type (resp. at time 1/2 in the second-type) process are identically distributed with the eigenvalues of Gaussian orthogonal (resp.unitary) random matrices. The second-type process is described by the stochastic differential equations of the Dyson-type Brownian motions with repulsive two-body forces proportional to the inverse of distances. The present study demonstrates that the spatio-temporal coarse graining of random walk models with contact interactions can provide many-body systems with long-range interactions.

PACS numbers: 05.40.-a, 02.50.Ey, 05.50.+q

The walks of independent random walkers, in which none of walkers have met others in a given time period, are called vicious walks. Since each random walk tends to a Brownian motion in the diffusion scaling limit, an interacting system of Brownian motions will be constructed as the scaling limit of vicious walkers [1, 2]. In an earlier paper, we explicitly performed the scaling limit of the vicious walkers in one dimension and derived the spatial-distribution function for the nonintersecting Brownian motions [3].

The purpose of this Letter is to study the scaling limit of vicious walkers as a stochastic process, while the previous paper reported its static properties at fixed times in order to clarify the relation with the timeless theory of Gaussian random matrices [4]. We first claim that the limit process is in general time-inhomogeneous, *i.e.* the transition probability depends on the time interval T, in which the nonintersecting condition is imposed. Then we study in detail two types of processes in time t defined by setting $T = t$ and $T \to \infty$, respectively. The former process is related with the Gaussian orthogonal ensemble (GOE) and the latter with the Gaussian unitary ensemble (GUE) in random matrix theory. This result demonstrates the fact that in the nonintersecting processes distributions at finite times depend on whether the nonintersecting condition will be also imposed in the future or not. We will derive the stochastic differential equations for the latter type and show that the drift terms act as the repulsive two-body forces proportional to the inverse of distances between particles. In other words, the scaling limit of vicious walkers can realize the Dyson-type Brownian motion models (at the inverse temperature $\beta = 2$) [5]. The Gaussian ensembles of random matrices can be regarded as the thermodynamical equilibrium of Coulomb gas system and that is the reason why Dyson introduced a one-dimensional model of interacting Brownian particles with (two-dimensional) Coulomb repulsive potentials. Here we should emphasize the fact that the vicious walkers, however, have only contact repulsive interactions to satisfy the nonintersecting condition. By taking the diffusion scaling limit, we can extract global effective interactions among walkers and they turn to be the long-ranged Coulomb-type interactions [6]. Such emergence of long-range effects in macroscopic scales from systems having only short-ranged microscopic interactions is found only at critical points in thermodynamical equilibrium systems, but it is a typical phenomenon enjoyed by a variety of interacting particle systems in far from equilibrium.

We also report the scaling limit of vicious walkers with a wall restriction. It will be shown that in this situation nonintersecting Brownian motions are discussed in relation to the *chiral* (Laguerre-type) Gaussian ensembles of random matrices [4, 7, 8].

Let $\{R_k^{s_j}\}_{k\geq 0}$, $j \in \{1, 2, \cdots, n\}$, be the n independent symmetric simple random walks on $\mathbf{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ started from n positions, $2s_1 < 2s_2 < \cdots < 2s_n, s_j \in \mathbf{Z}$. We fix the time interval as a positive even number m and impose the nonintersecting condition

$$R_k^{s_1} < R_k^{s_2} < \cdots < R_k^{s_n} \quad 1 \leq \forall k \leq m. \tag{1}$$

The subset of all possible random walks, which satisfy (1), is the vicious walks up to time m. For $2e_1 < 2e_2 < \cdots < 2e_n, e_j \in \mathbf{Z}$, let $V_n(\{R_k^{s_j}\}_{k=0}^m; R_m^{s_j} = 2e_j)$ be the probability that such vicious walks that the n walkers

arrive at the positions $\{2e_j\}_{j=1}^n$ at time m are realized in 2^{mn} random walks. We then consider, with a large value $L>0$, the rescaled lattice $\mathbf{Z}/(\sqrt{L}/2)$, where the unit length is $2/\sqrt{L}$, and let $\tilde{R}_k^x$ be the symmetric simple random walk started from x on this rescaled lattice. The following was proved in the previous paper [3]; for given $t>0$ and $0 \le x_1 < x_2 < \cdots < x_n, y_1 < y_2 < \cdots < y_n$, $\lim_{L\to\infty} V_n(\{\tilde{R}_k^{x_j}\}_{k=0}^{Lt}; \tilde{R}_{Lt}^{x_j} \in [y_j, y_j+dy_j]) = f_n(t;\{y_j\}|\{x_j\})d^n y$ with

$$f_n(t;\{y_j\}|\{x_j\}) = (2\pi t)^{-n/2} s_{\xi(y)}\left(e^{x_1/t},\cdots,e^{x_n/t}\right) \times e^{-\sum(x_j^2+y_j^2)/2t} \prod_{1\le j<k\le n}(e^{x_k/t}-e^{x_j/t}), \qquad (2)$$

where $s_\lambda(z_1,\cdots,z_n)$ is the Schur function [9] and $\xi_j(y) = y_{n-j+1}-(n-j)$ [10]. The function (2) shows the dependence on the initial positions of particles $\{x_j\}$ and the end-positions $\{y_j\}$ at time t of the probability that n independent one-dimensional Brownian particles do not intersect up to time t. Its summation over all end-positions is denoted by $\mathcal{N}_n(t;\{x_j\}) = \int_{y_1<\cdots<y_n} d^n y\, f_n(t;\{y_j\}|\{x_j\})$ and its asymptote for $t\to\infty$ was calculated as

$$\mathcal{N}_n(t;\{x_j\}) \simeq (2\pi)^{-n/2} t^{-\eta_n} b_n(\{x_j\})/c_n \qquad (3)$$

with $\eta_n = n(n-1)/4$, $b_n(\{x_j\}) = s_{\xi(x)}(1,\cdots,1) = \prod_{1\le j<k\le n}(x_k-x_j)/(k-j)$ and $c_n = n!(2^{3n/2}\prod_{j=1}^n \Gamma(j/2+1))^{-1}$, where $\Gamma(z)$ is the gamma function [3].

Since the vicious walkers are defined by imposing the nonintersecting condition (1) up to a given time m, the process depends on the choice of m. That is, the process is time-inhomogeneous. This feature is inherited in the process obtained in the diffusion scaling limit as explained below. Let $T>0$ and consider the n nonintersecting Brownian motions in the time interval $[0,T]$. For $0\le s<t\le T$, $x_1<\cdots<x_n, y_1<\cdots<y_n$, the transition probability densities from the configuration $\{x_j\}$ at time s to $\{y_j\}$ at t is given by

$$g_n^T(s,\{x_j\};t,\{y_j\}) = \frac{f_n(t-s;\{y_j\}|\{x_j\})\mathcal{N}_n(T-t;\{y_j\})}{\mathcal{N}_n(T-s;\{x_j\})}, \qquad (4)$$

since the numerator in RHS gives the nonintersecting probability for $[0,T]$ specified with the configurations $\{x_j\}$ and $\{y_j\}$ at times s and t, respectively, and the denominator gives the probability only specified with $\{x_j\}$ at s, where we have used the Markov property of the process. The time-inhomogeneity is obvious, since RHS depends not only $t-s$ but also $T-s$ and $T-t$.

In this Letter we study two special cases of T ; case (A) $T=t$ and case (B) $T\to\infty$. We will show that there is an interesting correspondence between these choices of T and the Gaussian ensembles of random matrices [4]. In order to see it we consider the limit $x_j\to 0$ $(1\le\forall j\le n)$.

Case (A) $T=t$. Since $\lim_{t\to 0} f_n(t;\{y_j\}|\{x_j\}) = \prod_{j=1}^n \delta(x_j-y_j)$ with Dirac's delta functions, $\mathcal{N}_n(0;\{x_j\})=1$ for any $\{x_j\}$, and setting $T=t$ makes (4) depend only on $t-s$ [11]. For $t>0$ and $|x|\equiv\sum_{j=1}^n |x_j| \ll 1$

$$\begin{aligned}\mathcal{N}_n(t;\{x_j\}) &= (2\pi t)^{-n/2}\prod_{1\le j<k\le n}(e^{x_k/t}-e^{x_j/t}) \\ &\times\int_{y_1<\cdots<y_n} d^n y\ s_{\xi(y)}(1,\cdots,1)e^{-\sum y_j^2/2t}(1+\mathcal{O}(|x|)) \\ &= \frac{t^{n(n-1)/4}}{(2\pi)^{n/2}c_n}\prod_{1\le j<k\le n}\frac{e^{x_k/t}-e^{x_j/t}}{k-j}\ (1+\mathcal{O}(|x|)).\end{aligned}$$

Then we have

$$g_n^t(0,\{0\};t,\{y_j\}) = c_n t^{-\zeta_n} e^{-\sum y_j^2/2t}\prod_{1\le j<k\le n}(y_k-y_j)$$

with $\zeta_n = n(n+1)/4$ for $y_1<\cdots<y_n$. If we set $t=1$ and assume $y_1<\cdots<y_n$, then

$$g_n^1(0,\{0\};1,\{y_j\}) = n!\ g_n^{\rm GOE}(\{y_i\}),$$

where $g_n^{\rm GOE}(\{y_i\})$ is the probability density of eigenvalues of GOE random matrices [12].

Case (B) $T\to\infty$. Let

$$p_n(s,\{x_j\};t,\{y_j\}) \equiv \lim_{T\to\infty} g_n^T(s,\{x_j\};t,\{y_j\}).$$

By virtue of (3) we can determine the explicit form for any initial configuration $\{x_j\}$ in this case as

$$p_n(0,\{x_j\};t,\{y_j\}) = \frac{h_n(\{y_j\})}{h_n(\{x_j\})} f_n(t;\{y_j\}|\{x_j\}), \qquad (5)$$

with the Vandermonde determinant $h_n(\{x_j\}) = \prod_{1\le j<k\le n}(x_k-x_j)$. In particular, if we take the limit $x_j\to 0$ $(1\le\forall j\le n)$, we have

$$p_n(0,\{0\};t,\{y_j\}) = c_n' t^{-\zeta_n'} e^{-\sum y_j^2/2t}\prod_{1\le j<k\le n}(y_k-y_j)^2, \qquad (6)$$

with $\zeta_n' = n^2/2$ and $c_n' = ((2\pi)^{n/2}\prod_{j=1}^{n-1} j!)^{-1}$. In this case we set $t=1/2$ and assume $y_1<\cdots<y_n$. Then

$$p_n(0,\{0\};1/2,\{y_j\}) = n!\ g_n^{\rm GUE}(\{y_i\}),$$

where $g_n^{\rm GUE}(\{y_i\})$ is the probability density of eigenvalues of GUE random matrices. In this case the nonintersecting condition will be imposed forever ($T\to\infty$), while in the case (A) there will be no condition in the future. The distributions of particles at finite times depend on the condition in the future.

By generalizing the calculation, which we did in the case (A), for arbitrary T and comparing the result with (6), we have

$$\begin{aligned}&\frac{g_n^T(0,\{0\};t,\{y_j\})}{p_n(0,\{0\};t,\{y_j\})} \\ &= \bar{c}_n T^{-n(n-1)/4} t^{n(n-1)/2}\frac{\mathcal{N}_n(T-t;\{y_j\})}{\prod_{1\le j<k\le n}(y_k-y_j)}\end{aligned}$$

with $\bar{c}_n = c_n/c'_n = \pi^{n/2} 2^{-n} \prod_{j=1}^{n} \Gamma(j+1)/\Gamma(j/2+1)$. This equality can be regarded as the multi-variable generalization of Imhof's relation between the probability distributions of Brownian meander and the Bessel process [13].

In the case (B) we have obtained the explicit expression (5) for arbitrary $\{x_j\}$ and we can derive a system of stochastic differential equations for the process. Using it we will explain why we have the GUE distribution at time $t = 1/2$. Let

$$a^k(\{x_j\}) = \sum_{j=1; j\neq k}^{n} \frac{1}{x_k - x_j} \quad \text{for } k = 1, 2, \cdots n.$$

It is easy to verify that

$$a^k(\{x_j\}) = \partial_k \log h_n(\{x_j\}), \tag{7}$$

$$\sum_{k=1}^{n} \left[\partial_k a^k(\{x_j\}) + \left(a^k(\{x_j\})\right)^2 \right] = 0, \tag{8}$$

where $\partial_k = \partial/\partial x_k$. Using these equalities, we will prove shortly that $p_n(0, \{x_j\}; t, \{y_j\})$ solves the diffusion equation

$$\frac{\partial}{\partial t} u(t; \{x_j\}) = \frac{1}{2} \Delta u(t; \{x_j\}) + \sum_{k=1}^{n} a^k(\{x_j\}) \partial_k u(t; \{x_j\}), \tag{9}$$

where $\Delta = \sum_{k=1}^{n} \partial_k^2$. This implies that the process defined in the case (B) is the system of n particles with positions $x_1(t), x_2(t), \cdots, x_n(t)$ at time t on the real axis, whose time evolution is governed by the stochastic differential equations

$$dx_k(t) = a^k(\{x_j(t)\})dt + dB_k(t), \quad k = 1, 2, \cdots, n, \tag{10}$$

where $\{B_k(t)\}_{k=1}^{n}$ are the independent standard Brownian motions; $B_j(0) = 0, \langle B_j(t) \rangle \equiv 0$ and $\langle (B_j(t) - B_j(s))(B_k(t) - B_k(s)) \rangle = |t - s| \delta_{jk}$. Because of the scaling property of Brownian motion, $\sqrt{\kappa} B_j(t)$ is equal to $B_j(\kappa t)$ in distribution. Then, if we set $t = 2t'$ and write $x_k(t) = \tilde{x}_k(t')$, (10) is equivalent with the $\alpha = 0, \beta = 2$ case of the equations $d\tilde{x}_k(t') = -\beta \partial_k W^{\alpha}(\{\tilde{x}_j(t')\})dt' + \sqrt{2} dB_k(t'), k = 1, 2, \cdots, n$, with $W^{\alpha}(\{x_j\}) = \alpha \sum_{j=1}^{n} x_j^2/2 - \sum_{1\leq j<k\leq n} \log(x_k - x_j)$. When $\alpha = 1$, they are known as the stochastic differential equations for the Dyson Brownian motions at the inverse temperature $\beta = 1/kT$ and the stationary distribution $\propto \exp(-\beta W^1(\{x_j\})$ [4, 5]. If $\alpha = 0$, the factor $\exp(-\beta \sum x_j^2/2)$ is replaced by $\exp(-\sum \tilde{x}_j^2/4t')$ for finite t' and thus when $t' = 1/(2\beta)$ we may have the Gaussian distribution with β. Setting $\beta = 2$ gives $t = 2t' = 1/2$. It should be noted that the system of diffusion equations describing the Dyson Brownian motions with $\beta = 2$ can be mapped to the free fermion model [6, 14].

Now we prove that (5) satisfies (9). First we remark that [3]

$$f_n(t; \{y_j\}|\{x_j\}) = \det_{1\leq j,k\leq n} \left((2\pi t)^{-1/2} e^{-(x_k - y_j)^2/2t} \right).$$

That is, f_n is a finite summation of the products of Gaussian kernels and thus it satisfies the diffusion equation [15]. Therefore

$$\frac{\partial}{\partial t} p_n(t; \{y_j\}|\{x_j\}) = \frac{1}{2} \frac{h_n(\{y_j\})}{h_n(\{x_j\})} \Delta f_n(t; \{y_j\}|\{x_j\}).$$

Then we can find that, if $\{a^k(\{x_j\})\}$ satisfy the equations

$$\sum_{k=1}^{n} a^k(\{x_j\}) \frac{1}{h_n(\{x_j\})} \partial_k f_n(t; \{y_j\}|\{x_j\}) = -\sum_{k=1}^{n} \partial_k(1/h_n(\{x_j\})) \partial_k f_n(t; \{y_j\}|\{x_j\}) \tag{11}$$

and

$$\sum_{k=1}^{n} a^k(\{x_j\}) \partial_k(1/h_n(\{x_j\})) = -\frac{1}{2} \sum_{k=1}^{n} \partial_k^2(1/h_n(\{x_j\})), \tag{12}$$

the proof will be completed. It is easy to see that (11) is satisfied if (7) holds for all k. Moreover, using (7), we can reduce (12) to (8). Then the proof is completed. This proof shows that the origin of the long-ranged interactions (7) is the normalization factor $1/h_n(\{x_j\})$ in (5).

In order to derive the chiral versions of the Gaussian ensembles of random matrices [4, 7, 8] and of the Dyson Brownian motions, next we consider the vicious walker problem with a wall restriction [16]. We impose the condition

$$R_k^{s_1} \geq 0 \quad 1 \leq \forall k \leq m \tag{13}$$

in addition to (1). That is, there assumed to be a wall at the origin and all walkers can walk only in the region $x \geq 0$. We write the probability to realize the vicious walks which satisfy these conditions (1), (13) and $\{R_m^{s_j} = 2e_j\}_{j=1}^{n}$ as $V_n^{+}(\{R_k^{s_j}\}_{k=0}^{m}; R_m^{s_j} = 2e_j)$. By the Lindström-Gessel-Viennot theorem and the reflection principle of random walks [16], we have

$$V_n^{+}(\{R_k^{s_j}\}_{k=0}^{m}; R_m^{s_j} = 2e_j) = 2^{-mn} \times \det_{1\leq j,k\leq n} \left(\binom{m}{\frac{m}{2} + s_k - e_j} - \binom{m}{\frac{m}{2} + s_k + (e_j + 1)} \right).$$

Following the same calculation as was done in [3], we will obtain, for $t > 0, 0 \leq x_1 < \cdots < x_n$, $0 \leq y_1 < \cdots < y_n$, $\lim_{L\to\infty} V_n^{+}(\{\tilde{R}_k^{x_j}\}_{k=0}^{Lt}; \tilde{R}_{Lt}^{x_j} \in [y_j, y_j + dy_j]) = f_n^{+}(t; \{y_j\}|\{x_j\}) d^n y$, with

$$f_n^{+}(t; \{y_j\}|\{x_j\}) = (2\pi t)^{-n/2} sp_{\xi^{+}(x)} \left(e^{y_1/t}, \cdots, e^{y_n/t} \right) \times e^{-\sum (x_j^2 + y_j^2)/2t} \prod_{j=1}^{n} (e^{y_j/t} - e^{-y_j/t}) \times \prod_{1\leq j<k\leq n} \{(e^{y_k/t} - e^{y_j/t})(e^{(y_j+y_k)/t} - 1)\},$$

where $sp_\lambda(z_1,\cdots,z_n)$ is the character of the irreducible representation specified by a partition $\lambda=(\lambda_1,\cdots,\lambda_n)$ of the symplectic Lie algebra (see, for example, Lectures 6 and 24 in [9]) given by $\det(z_j^{\lambda_k+n-k+1}-z_j^{-(\lambda_k+n-k+1)})/\det(z_j^{n-k+1}-z_j^{-(n-k+1)})$, and we set $\xi^+(x)=(\xi_1^+(x),\cdots,\xi_n^+(x))$ with $\xi_j^+(x)=x_{n-j+1}-(n-j+1)$. It is known that $sp_\lambda(1,\cdots,1)=\prod_{1\le j<k\le n}(\ell_k^2-\ell_j^2)/(m_k^2-m_j^2)\prod_{j=1}^n \ell_j/m_j$ with $\ell_j=\lambda_j+n-j+1, m_j=n-j+1$ [9] and

$$\int d^n u\ e^{-\sum u_j^2/2}\prod_{1\le j<k\le n}|u_k^2-u_j^2|^{2\gamma}\prod_{j=1}^n |u_j|^{2\alpha-1}$$
$$=2^{\alpha n+\gamma n(n-1)}\prod_{j=1}^n \frac{\Gamma(1+j\gamma)\Gamma(\alpha+\gamma(j-1))}{\Gamma(1+\gamma)}$$

for $\mathrm{Re}\,\alpha>0$ ((17.6.6) on page 354 in [4]). Then we can obtain the asymptote of $\mathcal{N}_n^+(t;\{x_j\})=\int_{0\le y_1<\cdots<y_n} d^n y f_n^+(t;\{y_j\}|\{x_j\})$ for $t\gg 1$ as

$$\mathcal{N}_n^+(t;\{x_j\})=t^{-n(2n+1)/2}(1+\mathcal{O}(1/t))\ \frac{b_n^+(\{x_j\})}{(2\pi)^{n/2}n!}$$
$$\times\int d^n y\ e^{-\sum y_j^2/2t}\prod_{1\le j<k\le n}|y_k^2-y_j^2|\prod_{j=1}^n |y_j|$$
$$\simeq t^{-\eta_n^+}b_n^+(\{x_j\})\frac{2^{n(n+2)/2}}{\pi^n n!}\ \prod_{j=1}^n \Gamma\left(1+\frac{j}{2}\right)\Gamma\left(1+\frac{j-1}{2}\right)$$

with $\eta_n^+=n^2/2$, where $b_n^+(\{x_j\})=\prod_{1\le j<k\le n}(x_k^2-x_j^2)/(k^2-j^2)\prod_{j=1}^n x_j/j$.

Two types of diffusion processes can be defined as in the cases without a wall. In the case (A) we will conclude that at time $t=1$ the positions of the nonintersecting Brownian particles all started from the origin with a wall restriction is identically distributed with the positive eigenvalues of random matrices in the chiral GOE. In the case (B) the transition probability density is given as

$$p_n^+(0,\{x_j\};t,\{y_j\})=\frac{h_n^+(\{y_j\})}{h_n^+(\{x_j\})}f_n^+(t;\{y_j\}|\{x_j\})$$

with $h_n^+(\{x_j\})=\prod_{1\le j<k\le n}(x_k^2-x_j^2)\prod_{j=1}^n x_j$. We can prove that $p_n^+(0,\{x_j\};t,\{y_j\})$ satisfies the diffusion equation (9) if $a^k(\{x_j\})$ is replaced by $a^{k+}(\{x_j\})=\sum_{j=1;j\neq k}^n 2x_k/(x_k^2-x_j^2)+1/x_k$. When $\{x_j\}=\{0\}$, the distribution at time $t=1/2$ may be equal to the distribution of the positive eigenvalues of random matrices in the chiral GUE.

In summary, we have studied the diffusion processes derived as the scaling limits of vicious walkers in one dimension with and without a wall. Two types of interacting Brownian particles were defined depending on the situations in the future and interesting correspondence between these processes and the Gaussian ensembles of random matrices was discussed. The systems of stochastic differential equations for the second-type processes were determined, which can be identified with the Dyson Brownian motion models at $\beta=2$ with appropriate change of time-scales. Further study of general time-inhomogeneous processes determined by (4) will be an interesting future problem.

The authors acknowledge useful discussions with T. Fukui, P. Forrester, T. Nagao and J. Cardy.

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